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# VIBRATIONS OF ORTHOTROPIC, RECTANGULAR PLATES OF NON-UNIFORM THICKNESS AND TWO ADJACENT FREE EDGES

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## 1. INTRODUCTION

The problem of transverse vibrations of a rectangular plate with two adjacent, free edges is a rather difficult elastomechanics problem since the governing boundary conditions at the free edges† (see Figure 1(a)),

$$M_x|_{x=a} = Q_x + \partial M_{xy}/\partial y|_{x=a} = 0, \qquad M_y|_{y=b} = Q_y + \partial M_{yx}/\partial x|_{y=b} = 0, \qquad \text{(1a, b)}$$

are difficult to satisfy exactly if one looks for an analytical solution.

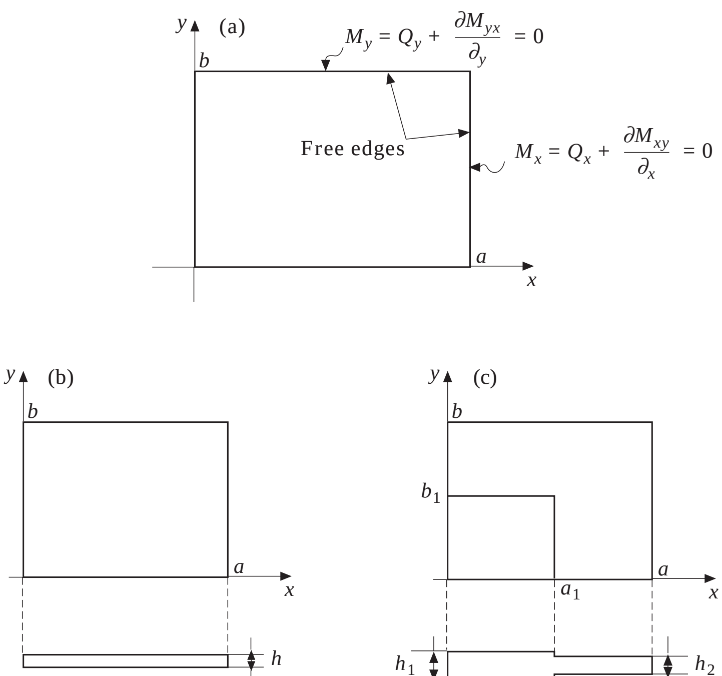


Figure 1. Rectangular plate with two adjacent, free edges: (a) governing boundary conditions at the free edges; (b) plate of uniform thickness; (c) plate of discontinuously varying thickness considered in the present study ($a_1/a = b_1/b$).

†Equations (1) are expressed in terms of stress resultants following well established notations e.g. $M_x$ denotes the bending moment acting in a plane parallel to the $xz$-plane, $Q_x$: shear force acting in the same plane as $M_x$, and $M_{xy}$ is the twisting moment in a plane parallel to the $y$–$z$ plane [1].

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A convenient approach followed in the literature consists in constructing co-ordinate functions which satisfy the boundary conditions at the edges $x = 0$ and $y = 0$ and second and third order derivatives with respect to $x$, equal to zero at $x = a$ and, similarly, at $y = b$. Very frequently these co-ordinate functions are the eigenfunctions of the Bernoulli–Euler vibrating beam problem and they are usually called "beam functions". Very accurate determinations of frequency coefficients are available in the case of isotropic plates [2].

On the other hand, very limited information is available in the case of orthotropic plates, the problem being of considerable importance in view of the use of orthotropic materials in a large variety of applications: from civil engineering to naval and ocean engineering applications to electronic packages where printed circuit boards are one of their fundamental elements.

The present study reports numerical experiments performed on the determination of the fundamental frequency of transverse vibration of orthotropic plates of uniform thickness, Figure 1(b); discontinuously varying thickness, Figure 1(c).

The frequency coefficients are determined using the optimized Rayleigh–Ritz method and a "pseudo" Fourier expansion which contains optimization parameters in the arguments of the sinusoidal terms [3, 4]. Once the determinantal equation is constructed, the lower root is minimized with respect to the optimization parameters. The procedure, essentially a non linear optimization process, constitutes an extension of the studies previously reported and where a single free edge was considered [3, 4]. The distribution of non-uniform thickness does not influence the formulation of the problem.

Good engineering agreement with accurate determinations of the fundamental frequency obtained by means of the finite element method is shown to exist [5, 6].

## 2. APPROXIMATE ANALYTICAL SOLUTION

The plate response is approximated using co-ordinate functions which satisfy the essential and natural boundary conditions at $x = 0$ and $y = 0$,

$$W(x, y) \simeq W_a(x, y) = \sum_j A_j X_j(x) Y_j(y). \tag{2}$$

Furthermore, $X_j$ and $Y_j$ contain optimization parameters which allow for the minimization of the fundamental eigenvalue. Using Lekhnitskii's standard notation [1] and substituting in the governing functional

$$f(W) = (\textit{Maximum strain energy}) - (\textit{Maximum kinetic energy})$$

$$J[W] = \tfrac{1}{2} \int\int \left[ D_1 \left( \frac{\partial^2 W}{\partial x^2} \right)^2 + 2D_1 \nu_2 \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} + D_2 \left( \frac{\partial^2 W}{\partial y^2} \right)^2 \right.$$

$$\left. + 4D_k \left( \frac{\partial^2 W}{\partial x \, \partial y} \right)^2 \right] \mathrm{d}x \, \mathrm{d}y - \frac{\rho \omega^2}{2} \int\int hW \, \mathrm{d}x \, \mathrm{d}y, \tag{3}$$

one generates a homogeneous, linear system of equations in the $A_j$'s, when equation (3) is minimized with respect to the $A_j$'s, $j = 1, 2. \ldots$ The non-triviality condition yields a secular determinant whose lowest root is the fundamental frequency coefficient.

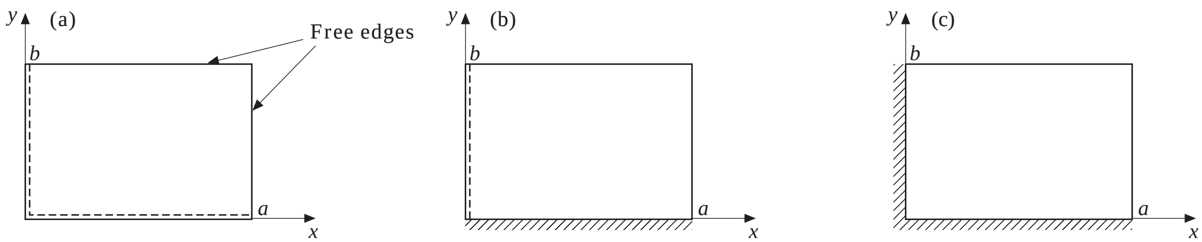


Figure 2. Different combinations of boundary conditions in the present study; (a) two adjacent edges simply supported; (b) edge $x = 0$ simply supported and $y = 0$ clamped; (c) two adjacent edges clamped.

$\Omega_1 = \sqrt{\rho h / D_1}\omega_1 a^2$. The following co-ordinate functions are used in the present investigation:

*Edges*: $x = 0$ and $y = 0$; simply supported (Figure 2(a)),

$$W \simeq W_a = \sum_{j=1,3}^{5} A_j \sin\frac{\pi x}{\gamma_j a} \sin\frac{\pi y}{\gamma_{j+1} b}, \qquad \gamma_1, \gamma_2 > 1. \tag{4}$$

*Edge*: $x = 0$; simply supported and $y = 0$ clamped (Figure 2(b)),

$$W \simeq W_a = \sum_{j=1,3}^{5} A_j \sin\frac{\pi x}{\gamma_j a} \sin^2\frac{\pi y}{\gamma_{j+1} b}, \qquad \gamma_1, \gamma_2 > 1. \tag{5}$$

*Edges*: $x = 0$ and $y = 0$; clamped (Figure 2(c)),

$$W \simeq W_a = \sum_{j=1,3}^{5} A_j \sin^2\frac{\pi x}{\gamma_j a} \sin^2\frac{\pi y}{\gamma_{j+1} b}, \qquad \gamma_1, \gamma_2 > 1. \tag{6}$$

since

$$\Omega_1 = \sqrt{\rho h / D_1}\omega_1 a^2 = \Omega_1(\gamma_1, \gamma_2, \ldots, \gamma_6) \tag{7}$$

by requiring

$$\partial\Omega_1/\partial\gamma_1 = \partial\Omega_1/\partial\gamma_2 = , \ldots, \partial\Omega_1/\partial\gamma_6 = 0, \tag{8}$$

one obtains an optimized value of $\Omega_1$. Admittedly, accomplishing this would require a non-linear optimization procedure but the following previous studies have been performed using a numerical searching process [3, 4].

## 3. FINITE ELEMENT SOLUTION

The present study makes use of the orthotropic plate element developed in reference [6] which is an extension of the well known isotropic plate element due to Bogner *et al*. [5].

## 4. NUMERICAL RESULTS

Table 1 presents values of $\Omega_1 = \sqrt{\rho h / D}\omega_1 a^2$ for rectangular isotropic plates of uniform thickness. The comparison with very accurate results obtained by Leissa [2] reveals that the present approach yields excellent engineering accuracy (when two adjacent edges are simply supported, the maximum difference occurs for $a/b = 1$ and is of the order of 0·3%; in the case of Figure 2(b) the maximum difference is approximately 1% and when the two adjacent edges are clamped the present approach yields a value of $\Omega_1$ which is about 4% higher than the eigenvalue determined in reference [2] for $a/b = 5/2$).

The convergence of the proposed approach is illustrated in Table 1 in the case of two adjacent; simply supported edges. Admittedly: the accuracy obtained by means of the present analytical procedure is always better in the case of this configuration. The minimization approach has been performed numerically in all cases. Table 2 depicts a comparison of values of $\Omega_1 = \sqrt{\rho h/D_1}\omega_1 a^2$ in the case of orthotropic plates of uniform thickness, between the analytical predictions and the results obtained by means of the finite element method. The maximum difference takes place in the case of Figure 2(c) for $a/b = 1$ and is of the order of 1·6%. Table 3 presents fundamental eigenvalues in the case of isotropic rectangular plates of discontinuously varying thickness. All calculations have been performed for $a_1/a = b_1/b = 1/2$, 3/4 and $h_1/h_2 = 2/3$. The agreement between

TABLE 1

*Fundamental frequency coefficient $\Omega_1$† in the case of isotropic, rectangular plates of uniform thickness with two adjacent free edges: comparison with values available in the literature*

| | | $a/b$ | | | | |
|---|---|---|---|---|---|---|
| Case | | 2/5 | 2/3 | 1 | 3/2 | 5/2 |
| Fig. 2(a) | Ref. [2] | 1·3201 | 2·2339 | 3·3687 | 5·0263 | 8·2506 |
| | 1 term | 1·3301 | 2·2671 | 3·4283 | 5·1011 | 8·3133 |
| | 2 terms | 1·3240 | 2·2499 | 3·4041 | 5·0623 | 8·2755 |
| | 3 terms | 1·3229 | 2·2396 | 3·3803 | 5·0391 | 8·2633 |
| Fig. 2(b) | Ref. [2] | 1·6160 | 3·0804 | 5·3639 | 9·9555 | 24·0887 |
| | 3 terms | 1·6271 | 3·1006 | 5·3832 | 9·9786 | 24·0990 |
| Fig. 2(c) | Ref. [2] | 3·9857 | 4·9848 | 6·9421 | 11·2160 | 24·2564 |
| | 3 terms | 4·0410 | 5·1100 | 6·9984 | 11·5381 | 25·2564 |

† Values of $\Omega_1 = \sqrt{\rho h/D}\omega_1 a^2$ ($\mu = 0{\cdot}30$)

TABLE 2

*Fundamental frequency coefficient $\Omega_1$† in the case of orthotropic, rectangular plate of uniform thickness with two adjacent, free edges: comparison with values obtained by means of the finite element (F.E.) method ($D_2/D_1 = 1/2$; $D_k/D_1 = 1/3$; $\nu_2 = 0{\cdot}30$)*

| | | $a/b$ | | | | |
|---|---|---|---|---|---|---|
| Figure | | 2/5 | 2/3 | 1 | 3/2 | 5/2 |
| Fig. 2(a) | 3 terms | 1·2789 | 2·1667 | 3·2721 | 4·9032 | 8·0609 |
| | F.E. | 1·2752 | 2·1525 | 3·2546 | 4·8784 | 8·0376 |
| | | (1425)‡ | (1025) | (625) | (1025) | (1425) |
| Fig. 2(b) | 3 terms | 1·4911 | 2·7600 | 4·6245 | 8·1107 | 18·2694 |
| | F.E. | 1·4758 | 2·7301 | 4·5842 | 8·0504 | 18·1661 |
| | | (1400) | (1000) | (600) | (984) | (1368) |
| Fig. 2(c) | 3 terms | 3·9301 | 4·7602 | 6·3602 | 9·6154 | 19·4125 |
| | F.E. | 3·8927 | 4·7005 | 6·2653 | 9·4825 | 19·1855 |
| | | (1344) | (960) | (576) | (960) | (1344) |

† Values of $\Omega_1 = \sqrt{\rho h/D_1}\omega_1 a^2$
‡ Number of degrees of freedom of the particular model.

TABLE 3

*Fundamental frequency coefficient $\Omega_1$† in the case of isotropic rectangular plates of discontinuously varying thickness, with two adjacent free edges* ($\mu = 0{\cdot}30$)

| $a_1/a = b_1 b$ | Figure | Method | $a/b$: 1 | 2/3 | 2/5 |
|---|---|---|---|---|---|
| 0·50 | 2(a) | 3 terms | 2·9283 | 1·9524 | 1·1729 |
| | | F.E. | 2·9156 | 1·9438 | 1·1650 |
| | 2(b) | 3 terms | 4·1996 | 2·4566 | 1·3365 |
| | | F.E. | 4·1408 | 2·4198 | 1·3186 |
| | 2(c) | 3 terms | 5·1101 | 3·6785 | 2·9191 |
| | | F.E. | 5·0212 | 3·6151 | 2·8819 |
| 0·75 | 2(a) | 3 terms | 3·4079 | 2·2699 | 1·3554 |
| | | F.E. | 3·3957 | 2·2606 | 1·3506 |
| | 2(b) | 3 terms | 5·1252 | 2·9824 | 1·6066 |
| | | F.E. | 5·0868 | 2·9559 | 1·5927 |
| | 2(c) | 3 terms | 6·3317 | 4·5563 | 3·6144 |
| | | F.E. | 6·2622 | 4·5109 | 3·5817 |

† Values of $\Omega_1 = \sqrt{\rho h_1/D}\omega_1 a^2$.

TABLE 4

*Fundamental frequency coefficient $\Omega_1$† in the case of orthotropic rectangular plates of discontinuously varying thickness, with two adjacent free edges* ($D_2 = 1/2D_1,\ D_k = 1/3\ D_1,\ \mu_2 = 0{\cdot}30$)

| $a_1/a = b_1 b$ | Figure | Method | $a/b$: 1 | 2/3 | 2/5 |
|---|---|---|---|---|---|
| 0·50 | 2(a) | 3 terms | 2·8533 | 1·9028 | 1·1432 |
| | | F.E. | 2·8388 | 1·8915 | 1·1320 |
| | 2(b) | 3 terms | 3·6649 | 2·2447 | 1·2719 |
| | | F.E. | 3·6063 | 2·2000 | 1·2350 |
| | 2(c) | 3 terms | 4·6994 | 3·5310 | 2·8791 |
| | | F.E. | 4·5965 | 3·4602 | 2·8307 |
| 0·75 | 2(a) | 3 terms | 3·3174 | 2·2095 | 1·3180 |
| | | F.E. | 3·3030 | 2·1972 | 1·3135 |
| | 2(b) | 3 terms | 4·4582 | 2·7053 | 1·5087 |
| | | F.E. | 4·4153 | 2·6727 | 1·4815 |
| | 2(c) | 3 terms | 5·8251 | 4·3863 | 3·5695 |
| | | F.E. | 5·7462 | 4·3330 | 3·5263 |

† Values of $\Omega_1 = \sqrt{\rho h_1/D_1}\omega_1 a^2$.

the analytical predictions and the results obtained using the finite element method is very good from an engineering viewpoint, the maximum difference takes place for the configuration shown in Figure 2(c) and is of the order of 2% for $a/b = 2/3$.

Table 4 shows values of $\Omega_1$ for orthotropic rectangular plates of discontinuously varying thickness and for the same values of $a_1/a$, $b_1/b$ and $h_2/h_1$ as were used for the isotropic structure. The agreement between the analytical and the FE predictions is, again, quite satisfactory.

In conclusion, one may say that the proposed analytical approach is quite simple† and allows for the determination of the fundamental frequency of transverse vibration of a rather complicated structural element with sufficient accuracy.

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†Admittedly: the procedure is rather tedious but it is greatly simplified by the use of MATHEMATICA.